Quantitative Genomics and Genetics
BTRY 4830/6830; PBSB.5201.01

Lecture 18: Introduction to Covariates (Population structure) and QQ plots

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April 9, 2020 (Th) 8:40-9:55
• Logistics (!!!)

• **The Midterm will be available (CMS) 7AM on Mon., April 13 and due (CMS) before 7:01 PM Tues., April 14 (!!)**

• Computer lab will be held today / tomorrow (help sessions) April 9 & 10 - please see your TA Piazza messages for instructions! - **NOTE THAT WE NOT LONGER TAKE ATTENDANCE BUT YOU WILL STILL HAVE QUIZZES!**

• NO CLASS TUES (April 14)!

• NO OFFICE HOURS Mon. (April 13)!

• Office hours (by zoom!) will be this Fri. (April 10):
  • 10AM-noon (same office hours zoom link as always)!
  • 3:30PM-5:30PM (same office hours zoom link as always)!
Announcements

• Changing your grade options (so far):
  • Cornell Ithaca: you have until May 12 to change your grade from letter to S/U or drop the class without a “W”
  • Cornell Tech: you have until April 21 to change your grade to S/U
  • Weill Cornell: no official guidance yet…

• Your grade for the class: mid-term (20%), final (30%), Problem Sets (20%), Project (25%), Computer Lab (5% - assignments!)
Quantitative Genomics and Genetics - Spring 2020
BTRY 4830/6830; PBSB 5201.01

Midterm - available 7 AM, Mon., April 13

Midterm exam due before 7:01 PM, Tues., April 14

PLEASE NOTE THE FOLLOWING INSTRUCTIONS:

1. You are to complete this exam alone. The exam is open book, so you are allowed to use any books or information available online, your own notes and your previously constructed code, etc. HOWEVER YOU ARE NOT ALLOWED TO COMMUNICATE OR IN ANY WAY ASK ANYONE FOR ASSISTANCE WITH THIS EXAM IN ANY FORM (the only exceptions are Rachel, Scott, and Dr. Mezey). As a non-exhaustive list this includes asking classmates or ANYONE else for advice or where to look for answers concerning problems, you are not allowed to ask anyone for access to their notes or to even look at their code whether constructed before the exam or not, etc. You are therefore only allowed to look at your own materials and materials you can access on your own. In short, work on your own! Please note that you will be violating Cornell’s honor code if you act otherwise.
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2. Please pay attention to instructions and complete ALL requirements for ALL questions, e.g. some questions ask for R code, plots, AND written answers. We will give partial credit so it is to your advantage to attempt every part of every question.

3. A complete answer to this exam will include R code answers in Rmarkdown, where you will submit your .Rmd script and associated .pdf file. Note there will be penalties for scripts that fail to compile (!!!). Also, as always, you do not need to repeat code for each part (i.e., if you write a single block of code that generates the answers for some or all of the parts, that is fine, but do please label your output that answers each question!!). You should include all of your plots and written answers in this same .Rmd script with your R code.

4. The exam must be uploaded on CMS before 7:01 PM, Tues., April 14. It is your responsibility to make sure that it is in uploaded by then and no excuses will be accepted (power outages, computer problems, Cornell’s internet slowed to a crawl, etc.). Remember: you are welcome to upload early! We will deduct points for being late for exams received after this deadline (even if it is by minutes!!).
• Last lecture we began our discussion of statistical and experimental issues that can impact GWAS

• Today we will complete this discussion by introducing the concept of covariate modeling and discuss a critical diagnostic: Quantile-Quantile (QQ) plots
Conceptual Overview

Genetic System

Does A1 → A2 affect Y?

Sample or experimental pop

Measured individuals (genotype, phenotype)

Regression model

Pr(Y|X)

Reject / DNR

Model params F-test
Review: Successful mapping of causal polymorphisms in GWAS

- For GWAS, we are generally concerned with correctly identifying the position of as many causal polymorphisms as possible (True Positives) while minimizing the number of cases where we identify a position where we think there is a causal polymorphism but there is not (False Positive).

- We are less concerned with cases where there is a causal polymorphism but we do not detect it (why is this?).

- Issues that affect the number of True Positives and False Positives that we identify in a GWAS can be statistical and experimental (or a combination).
Review: Statistical Issues: Multiple Testing

• Recall that when we perform a GWAS, we perform $N$ hypothesis tests (where $N$ is the number of measured genotype markers)

• Also recall that if we set a Type I error to a level (say 0.05) this is the probability of incorrectly rejecting the null hypothesis

• If we performed $N$ tests that were independent, we would therefore expect to incorrectly reject the null $N \times 0.05$ and if $N$ is large, we would therefore make LOTS of errors (!!)

• This is the multiple testing problem = the more tests we perform the greater the probability of making a Type I error

• Now in a GWAS, our tests are not independent (LD!) but we could still make many errors by performing $N$ tests if we do not set the Type I error appropriately
Review: Multiple test correction I

• A Bonferroni correction sets the Type I error for the entire GWAS using the following approach: for a desired type I error $\alpha$, set the Bonferroni Type I error $\alpha_B$ to the following:

$$\alpha_B = \frac{\alpha}{N}$$

• We therefore use the Bonferroni Type I error to assess EACH of our $N$ tests in a GWAS.

• For example, if we have $N=100$ in our GWAS and we want an overall GWAS Type I error of 0.05, we require a test to have a p-value less than 0.0005 to be considered significant.
Review: Multiple test correction II

- A False Discovery Rated (FDR) based approach (there are many variants!) uses the expected number of false positives to set (=control) the type 1 error.

- For $N$ tests and a specified Type 1 error, the FDR is defined in terms or the number of cases where the null hypothesis is rejected $R$:

  $$FDR = \frac{N \times \alpha}{R}$$

- Intuitively, the FDR is the proportion of cases where we reject the null hypothesis that are false positives.

- We can estimate the FDR for a GWAS, e.g. say for $N=100,000$ tests and a Type 1 error of 0.05, we reject the null hypothesis 10,000 times, the FDR $= 0.5$.

- FDR methods for controlling for multiple tests (e.g. Benjamini-Hochberg) set the Type 1 error to control the FDR to a specific level, say FDR=0.01 (what is the intuition at this FDR level?)
Review: Multiple test correction III

- Since the lower the Type I error the lower the power of our test, if we set the Type I error too low due to a very large N, we might not get any hits even when there are clear causal polymorphisms (is this desirable!?)

- In general, a Bonferroni correction sets a lower overall GWAS Type I error than FDR approaches (what are the trade-offs and why would we choose one over the other?)

- Both Bonferroni and FDR approaches make the implicit assumption that all tests are independent (which we know not to be the case in GWAS!)

- A strategy that can produce a more accurate Bonferroni or FDR cutoff is to use a permutation approach (which we do not have time to cover in this course)

- Regardless of the approach, some correction for multiple tests is necessary to guard against a case where there are no true positives in the experiment, i.e. this is why we do not automatically assume the highest “peak” is a true positive (unless it is significant after a multiple test correction)
Experimental issues that produce false positives

- Type 1 errors can produce a false positives (= places we identify in the genome as containing a causal polymorphism / locus that do not)

- However, there are experimental reasons why we can correctly reject the null hypothesis (= we do not make a Type 1 error) but we still get a false positive:
  - Cases of disequilibrium when there is no linkage
  - Genotyping errors
  - Unaccounted for covariates (stay tuned!)
  - There are others...
Statistical / experimental issues that affect True Positives: power I

• Recall that *power* is defined as the probability of correctly rejecting the null hypothesis when it is false (incorrect)

• Also recall that we cannot control power directly because it depends on the true parameter value(s) that we do not know!

• Also recall that we can indirectly control power by setting our Type I error, where there is a trade-off between Type I error and power (what is this trade-off!?)

• There are also a number of issues that affect power that are a function of the GWAS experiment
Statistical / experimental issues that affect True Positives: power II

- Power tends to increase with the increasing size of the true effect of the genotype on phenotype (how is this quantified in terms of linear regression parameters?)

- Power tends to increase with increasing sample size $n$

- Power tends to increase as the Minor Allele Frequency (MAF) increases (why is this?)

- Power tends to increase as the LD between a causal polymorphism and the genotype marker being tested increases (i.e. as the correlation between the causal and marker genotype increase)

- Power also depends on other factors including the type of statistical test applied, etc.

- Can any of these be controlled?
An issue specific to GWAS: resolution

- **Resolution** - the region of the genome indicated by significant tests for a set of correlated markers in a GWAS

- Recall that we generally consider a set of contiguous significant markers (a “skyscraper” on a Manhattan plot) to indicate the location of a single causal polymorphism (although it need not indicate just one!)

- Note that the marker with the most significant p-value within a set is not necessarily closest to the causal polymorphism (!!)

- In practice, we often consider a set of markers with highly significant p-values to span the region where a causal polymorphism is located (but this is arbitrary and need not be the case!)

- In general, resolution in a GWAS is limited by the level of LD, which means there is a trade-off between resolution and the ability to map causal polymorphisms and that there is a theoretical limit to the resolution of a GWAS experiment (what is this limit?)
Rigorous formulation of GWAS analysis

- For a GWAS, we assume that there could be causal polymorphisms $X = (X_a, X_d)$ that are BOTH in the same physical position of the genome AND are correlated (= in linkage disequilibrium) with polymorphisms that we have measured $X' = (X_a', X_d')$:

  $$|\text{Corr}(X, X')| >> 0$$

- Note we are using $|\text{Corr}(X, X')|$ and not specifying $X_a'$, $X_d'$ because either or both $X_a'$, $X_d'$ could be correlated with $X_a$, $X_d$

- For analysis of a GWAS with $N$ measured genotypes (2 alleles each) and a normal (error) phenotype we perform $N$ hypothesis tests:

  $$H_0 : \beta'_a = 0 \cap \beta'_d = 0$$
  $$H_A : \beta'_a \neq 0 \cup \beta'_d \neq 0$$

  $$Y = \beta'_0 + X'_a \beta'_a + X'_d \beta'_d + \epsilon$$

- For genotypes / sets of genotypes (which sets?) for which we reject the null, we assume that this indicates a position of causal polymorphism (we have mapped the position)
Experimental issues that produce false positives

- Type 1 errors can produce a false positives (= places we identify in the genome as containing a causal polymorphism / locus that do not)

- However, there are experimental reasons why we can correctly reject the null hypothesis (= we do not make a Type 1 error) but we still get a false positive:
  - Cases of disequilibrium when there is no linkage
  - Genotyping errors
  - **Unaccounted for covariates (stay tuned!)**
  - There are others...
Introduction to covariates I

- Recall that in a GWAS, we are considering the following regression model and hypotheses to assess a possible association for every marker with the phenotype

\[ Y = \beta_\mu + X_a\beta_a + X_d\beta_d + \epsilon \]

\[ H_0 : \beta_a = 0 \cap \beta_d = 0 \]

\[ H_A : \beta_a \neq 0 \cup \beta_d \neq 0 \]

- Also recall that with these hypotheses we are actually testing:

\[ H_0 : Cov(Y, X_a) = 0 \cap Cov(Y, X_d) = 0 \]

\[ H_A : Cov(Y, X_a) \neq 0 \cup Cov(Y, X_d) \neq 0 \]
Introduction to covariates II

- Let's consider these two cases:

- For the first, the marker is not correlated with a causal polymorphism but the factor is correlated with BOTH the phenotype and the marker such that a test of the marker using our framework will produce a false positive (!!):

\[
Cov(Y, X_z) \neq 0, \quad H_0 : \beta_a = 0 \cap \beta_d = 0
\]

\[
Cov(X_a, X_z) \neq 0, \quad H_A : \beta_a \neq 0 \cup \beta_d \neq 0
\]

\[
Y = \beta_\mu + X_a\beta_a + X_d\beta_d + \epsilon
\]

- For the second, the marker is correlated with a causal polymorphism and while the factor is correlated with the phenotype but not the marker, a test of the marker in our framework will model the effect of the factor in our error term (which will reduce power)!

\[
Cov(Y, X_z) \neq 0.
\]

\[
Cov(X_a, X_z) = 0
\]

\[
Y = \beta_\mu + X_a\beta_a + X_d\beta_d + \epsilon_{X_z}
\]

\[
\epsilon_{X_z} = X_z\beta_z + \epsilon
\]

\[
\epsilon \sim N(0, \sigma_\epsilon^2)
\]
Modeling covariates I

• Therefore, if we have a factor that is correlated with our phenotype and we do not handle it in some manner in our analysis, we risk producing false positives AND/OR reduce the power of our tests!

• The good news is that, assuming we have measured the factor (i.e. it is part of our GWAS dataset) then we can incorporate the factor in our model as a covariate(s):

\[ Y = \beta_\mu + X_a\beta_a + X_d\beta_d + X_{z,1}\beta_{z,1} + X_{z,2}\beta_{z,2} + \epsilon \]

• The effect of this is that we will estimate the covariate model parameter and this will account for the correlation of the factor with phenotype (such that we can test for our marker correlation without false positives / lower power!)
Modeling covariates II

- How do we perform inference with a covariate in our regression model?

- We perform MLE the same way (!!) our $X$ matrix now simply includes extra columns, one for each of the additional covariates, where for the linear regression we have:

$$MLE(\hat{\beta}) = (x^T x)^{-1} x^T y$$

- We perform hypothesis testing the same way (!!) with a slight difference: our LRT includes the covariate in both the null hypothesis and the alternative, but we are testing the same null hypothesis:

$$H_0 : \beta_a = 0 \cap \beta_d = 0$$

$$H_A : \beta_a \neq 0 \cup \beta_d \neq 0$$
Modeling covariates IV

- First, determine the predicted value of the phenotype of each individual under the null hypothesis (how do we set up $\mathbf{x}$?):
  \[
  \hat{y}_{i,\hat{\theta}_0} = \hat{\beta}_\mu + \sum_{j=1} x_{i,z,j} \hat{\beta}_{z,j}
  \]

- Second, determine the predicted value of the phenotype of each individual under the alternative hypothesis (set up $\mathbf{x}$?):
  \[
  \hat{y}_{i,\hat{\theta}_1} = \hat{\beta}_\mu + x_{i,a} \hat{\beta}_a + x_{i,d} \hat{\beta}_d + \sum_{j=1} x_{i,z,j} \hat{\beta}_{z,j}
  \]

- Third, calculate the “Error Sum of Squares” for each:
  \[
  SSE(\hat{\theta}_0) = \sum_{i=1}^n (y_i - \hat{y}_{i,\hat{\theta}_0})^2 \quad SSE(\hat{\theta}_1) = \sum_{i=1}^n (y_i - \hat{y}_{i,\hat{\theta}_1})^2
  \]

- Finally, we calculate the F-statistic with degrees of freedom [(2, n-#(alt hyp parameters)] (why two degrees of freedom?):
  \[
  F_{[2,n-\#(\hat{\theta}_1)}(\mathbf{y}, \mathbf{x}_a, \mathbf{x}_d) = \frac{SSE(\hat{\theta}_0) - SSE(\hat{\theta}_1)}{2} \cdot \frac{SSE(\hat{\theta}_1)}{n-\#(\hat{\theta}_1)}
  \]
Modeling covariates V

- Thus, for testing the null hypothesis in a linear regression, we can construct an F-test using a slightly different formula:

\[
SSE(\hat{\theta}_0) = \sum_{i=1}^{n} (y_i - \hat{y}_i,\hat{\theta}_0)^2
\]

\[
SSE(\hat{\theta}_1) = \sum_{i=1}^{n} (y_i - \hat{y}_i,\hat{\theta}_1)^2
\]

\[
F_{[2,n-\#(\hat{\theta}_1)]}(y, x_a, x_d) = \frac{SSE(\hat{\theta}_0) - SSE(\hat{\theta}_1)}{\frac{SSE(\hat{\theta}_1)}{n-\#(\hat{\theta}_1)}}
\]

- For the null hypotheses we are testing, once you calculate this F-statistic, you compare to an F-distribution with 2 and n - #(alternative hypothesis parameters) degrees of freedom.

- The "2" df in the numerator comes from the #(alternative hypothesis model parameters) - #(null hypothesis model parameters).

- Note that our previous formula for an F-statistic can be represented this way as well (!!)
Modeling covariates VI

• Say you have GWAS data (a phenotype and genotypes) and your GWAS data also includes information on a number of covariates, e.g. male / female, several different ancestral groups (different populations!!), other risk factors, etc.

• First, you need to figure out how to code the $X_z$ in each case for each of these, which may be simple (male / female) but more complex with others (where how to code them involves fuzzy rules, i.e. it depends on your context!!)

• Second, you will need to figure out which to include in your analysis (again, fuzzy rules!) but a good rule is if the parameter estimate associated with the covariate is large (=significant individual p-value) you should include it!

• There are many ways to figure out how to include covariates (again a topic in itself!!) - next lecture we will provide an (important!) example: population structure
We will now introduce an essential tool for detecting the most problematic covariates (and can be used to diagnose many other problems!): a Quantile-Quantile (QQ) plot.

While the definition of a QQ-plot is complex, you will see that how we generate a QQ-plot is easy!

We will demonstrate the value of a QQ plot for detecting the often problematic variable: population structure.

In general, whenever you perform a GWAS, you should construct a QQ plot (!!) and always include a QQ plot in your publication.
Quantile-Quantile (QQ) plots II

• Consider a random variable with a continuous probability distribution

• **quantile** - regular, equally spaced intervals of a random variable that divide the random variable into units of equal distribution

• A Quantile-Quantile (QQ) plot (in general) plots the observed quantiles of one distribution versus another OR plots the observed quantiles of a distribution versus the quantiles of the ideal distribution

• We will use a QQ plot to plot our the quantile distribution of observed p-values (on the y-axis) versus the quantile distribution of expected p-values (what distribution is this!?)
Quantile-Quantile (QQ) plots III

• How to construct a QQ plot for a GWAS:

  • If you performed $N$ tests, take the -log (base 10) of each of the p-values and put them in rank order from smallest to largest

  • Create a vector of $N$ values evenly spaces from 1 to $1 / N$ (how do we do this?), take the -log of each of these values and rank them from smallest to largest

  • Take the pair of the smallest of values of each of these lists and plot a point on an x-y plot with the observed -log p-value on the y-axis and the spaced -log value on the x-axis

  • Repeat for the next smallest pair, for the next, etc. until you have plotted all $N$ pairs in order
Quantile-Quantile (QQ) plots III

- In an ideal GWAS case where there ARE NO causal polymorphisms, your QQ plot will be a line:

- The reason is that we will observe a uniform distribution of p-values from such a case and in our QQ we are plotting this observed distribution of p-value versus the expected distribution of p-values: a uniform distribution (where both have been -log transformed)

- Note that if you GWAS analysis is correct but you do not have enough power to detect positions of causal polymorphisms, this will also be your result (!!), i.e. it is a way to assess whether you can detect any hits in your GWAS (!!)
Quantile-Quantile (QQ) plots IV

- In an ideal GWAS case where there ARE causal polymorphisms, your QQ plot will be a line with a tail (!!!):

- This happens because most of the p-values observed follow a uniform distribution (i.e. they are not in LD with a causal polymorphism so the null hypothesis is correct!) but the few that are in LD with a causal polymorphism will produce significant p-values (extremely low = extremely high \(-\log(p-values)\)) and these are in the “tail”

- This is ideally how you want your QQ-plot to look - if it does, you are in good shape!
Quantile-Quantile (QQ) plots

- In practice, you can find your QQ plot looks different than either the “null GWAS” case or the “ideal GWAS” case, for example:

- This indicates that something is wrong (!!!!) and if this is the case, you should not interpret any of your significant p-values as indicating locations of causal polymorphisms (!!!!)

- Note that this means that you need to find an analysis strategy such that the result of your GWAS produces a QQ plot that does NOT look like this (note that this takes experience and many tools to do consistently!)

- Also note that unaccounted for covariates can cause this issue and the most frequent culprit is unaccounted for population structure
That’s it for today

• Good luck on the exam! See you Thurs.!