Problem 1 (Easy)

Consider a coin (i.e., a system) and a two flip experiment, such that the sample space is:

Ω = \{HH, HT, TH, TT\}

Let’s define a function that maps sets of the sigma algebra to the following values:

\[

g(\emptyset) = 0 \\
g(\{HH\}) = 0.25, g(\{HT\}) = 0.25, g(\{TH\}) = 0.25, g(\{TT\}) = 0.25 \\
g(\{HH, HT\}) = 0.5, g(\{HH, TH\}) = 0.5, g(\{HH, TT\}) = 1.0 \\
g(\{HT, TH\}) = 0, g(\{HT, TT\}) = 0.5, g(\{TH, TT\}) = 0.5 \\
g(\{HH, TH, TT\}) = 0.75, g(\{HH, HT, TT\}) = 0.75 \\
g(\{HH, HT, TH\}) = 0.75, g(\{HH, HT, TT\}) = 0.75 \\
g(\{HH, HT, TH, TT\}) = 1.0
\]

a. Which specific components of this function violate the Axioms of Probability?

b. If we were to attempt to interpret this function as a probability function, explain what two distinct ‘probabilities’ each of the possible experimental outcomes HH, HT, TH, TT would have and how this would contradict our intuition about how probability is supposed to work?.

c. Write out a new function on this sigma algebra which leaves the two components of the current function that violate axioms of probability as they are but changes all of the other components so that the new function is a probability function.

d. For this new probability function, what is the chance that an experiment will produce HH, HT, TH, and TT? Could this probability model be the ‘correct’ model for a two flip coin experiment (explain your answer)?
Problem 2 (Medium)

Consider a coin (a system) and a two flip experiment. Assume a probability function on the sigma algebra of the sample space, where for the sets \{HH\}, \{HT\}, \{TH\}, \{TT\} this function assigns a probability of 0.25 to each.

a. Starting with this system, experiment, and probability model, what would be the new sample space if we KNOW that the first flip is T (i.e., stated another way, the first flip cannot be H)? What is the sigma algebra of this new sample space?

b. Starting from the original probability model (i.e., \(\Pr(\{HH\})=\Pr(\{HT\})=\Pr(\{TH\})=\Pr(\{TT\})=0.25\)) and using the definition of conditional probability, write out the probability function for the sigma algebra in [a].

c. Next assume we DO NOT know (i.e., we have NO information about) what the first or second flip of the experiment will be (i.e., NOT the case we considered in [a] and [b]). Consider a random variable \(X_1\) that takes the value ‘1’ if the first flip is H and that takes the value ‘0’ if the first flip is T. Write out how \(X_1\) assigns values to each element HH, HT, TH, TT of the sample space. Also write out the probabilities that \(X_1 = 0, X_1 = 1\).

d. Assume we DO NOT know (i.e., we have NO information about) what the first or second flip of the experiment will be (i.e., NOT the case we considered in [a] and [b]). Consider a random variable \(X_2\) that takes the value ‘1’ if the two flips are the same and ‘0’ if the two flips are different. Write out how \(X_2\) assigns values to each element HH, HT, TH, TT of the sample space. Also write out the probabilities that \(X_2 = 0, X_2 = 1\).

e. Assume we DO NOT know (i.e., we have NO information about) what the first or second flip of the experiment will be (i.e., NOT the case we considered in [a] and [b]). Consider a random variable \(X_3\) that takes the value ‘1’ if either of two flips are T and ‘0’ if neither of the two flips are T. Write out how \(X_3\) assigns values to each element HH, HT, TH, TT of the sample space. Also write out the probabilities that \(X_3 = 0, X_3 = 1\).

f. Are \(X_1\) and \(X_2\) independent? Demonstrate that this is the case.

g. Are \(X_1\) and \(X_3\) independent? Demonstrate that this is the case.

h. Are \(X_2\) and \(X_3\) independent? Demonstrate that this is the case.

i. Assume that \(X_1 = 1\). Show the probability model for \(X_2|X_1 = 1\), i.e., use the equation for conditional probability to calculate \(\Pr(X_2 = 0|X_1 = 1)\) and \(\Pr(X_2 = 1|X_1 = 1)\). Is this probability model the same as for \(X_2\), i.e., \(\Pr(X_2 = 0)\) and \(\Pr(X_2 = 1)\)? Why did you know this would be the case given your answer to [f]?

j. Assume that \(X_1 = 1\). Show the probability model for \(X_3|X_1 = 1\), i.e., use the equation for conditional probability to calculate \(\Pr(X_3 = 0|X_1 = 1)\) and \(\Pr(X_3 = 1|X_1 = 1)\). Is this probability model the same as for \(X_3\), i.e., \(\Pr(X_3 = 0)\) and \(\Pr(X_3 = 1)\)? Why did you know this would be the case given your answer to [g]?
Problem 3 (Difficult)

Consider the sample space and Sigma Algebra in question [1]. This is not the only Sigma Algebra that can be defined for this sample space. Write down another Sigma Algebra for this sample space and show that its sets satisfy the three properties of a Sigma Algebra. Also provide an intuitive explanation as to why this alternative Sigma Algebra you have presented cannot be used to define a function (model) for the experiment.