I am lecturing from NYC today and Ithaca Tues. (March 8)

Homework #3 is due 11:59PM March 8!

We will have office hours Mon. (March 7) 4:30-6:30 (I will send zoom information this weekend or Mon.)
Summary of lecture 11: hypothesis testing and genetic modeling

• Last lecture, we began our discussion of Hypothesis Testing
• Today, we will complete our discussion of Hypothesis Testing and being our introduction to Genetic Modeling!
Conceptual Overview

System

Experiment

Question

Sample

Inference

Prob. Models

Statistics

Assumptions
To use sample spaces in probability, we need a way to map these sets to the real numbers. We are going to define a mathematical operator that takes an input and produces an output.

\[
\begin{align*}
    \text{Random Variable} & \quad X(\omega), \omega \in \Omega \\
    \text{Experiment} & \quad \Omega \\
    \text{(Sample Space)} & \quad \mathcal{F} \\
    \text{(Sigma Algebra)} & \\
    \text{Statistical Sampling} & \quad \Pr(T(X)) \\
    \text{Distribution:} & \\
    \Pr([X_1 = x_1, \ldots, X_n = x_n]) & \\
    \Pr(X) & \\
    \text{Statistic:} & \quad T(x) \\
    [X_1 = x_1, \ldots, X_n = x_n] &
\end{align*}
\]
Estimators

**Estimator**: \( T(x) = \hat{\theta} \)

**Estimator (Statistic)**: \( P_r(T(X)|\theta), \theta \in \Theta \)

**Sampling Distribution**: \( P_r([X_1 = x_1, \ldots, X_n = x_n]) \)

\( X = x \)

\( P_r(X) \)

\( X = x \)

Random Variable

\( X(\omega), \omega \in \Omega \)

\( P_r(\mathcal{F}) \)

Experiment

\( \Omega \)

(Sample Space)

\( \mathcal{F} \)

(Sigma Algebra)
Hypothesis Tests

Hypothesis: $T(x)$, $H_0: \theta = c$

Statistic Sampling Distribution: $Pr(T(X)|\theta)$, $\theta \in \Theta$

$[X_1 = x_1, \ldots, X_n = x_n]$

$Pr([X_1 = x_1, \ldots, X_n = x_n])$

$X = x$

$Pr(X)$

$X$

Random Variable

$\mathcal{X}$

$X(\omega), \omega \in \Omega$

$Pr(\mathcal{F})$

Experiment

$\Omega$

(Sample Space)

$\mathcal{F}$

(Sigma Algebra)
Review: Hypothesis testing

- To build this framework, we need to start with a definition of hypothesis

- **Hypothesis** - an assumption about a parameter

- More specifically, we are going to start our discussion with a *null hypothesis*, which states that a parameter takes a specific value, i.e. a constant

\[ H_0 : \theta = c \]

- For example, for our height experiment / identity random variable, we have \( Pr(X|\theta) \sim N(\mu, \sigma^2) \) and we could consider the following null hypothesis:

\[ H_0 : \mu = 0 \]
Review: p-value I

- We quantify our intuition as to whether we would have observed the value of our statistics given the null is true with a \textit{p-value}.

- \textbf{p-value} - the probability of obtaining a value of a statistic \(T(x)\), or more extreme, conditional on \(H_0\) being true.

- Formally, we can express this as follows:

\[
pval = Pr(|T(x)| \geq t|H_0 : \theta = c)
\]

- Note that a p-value is a function on a statistic (!!) that takes the value of a statistic as input and produces a p-value as output in the range \([0, 1]\):

\[
pval(T(x)) : T(x) \rightarrow [0, 1]
\]
Review: p-value II

- More technically a p-value is determined not just by the probability of the statistic given the null hypothesis is true, but also whether we are considering a “one-sided” or “two-sided” test

- For a one-sided test (towards positive values), the p-value is:

\[
pval(T(x)) = \int_{T(x)}^{\infty} Pr(T(x) \mid \theta = c) dT(x)
\]

\[
pval(T(x)) = \max(T(X)) \sum_{T(x)} Pr(T(x) \mid \theta = c)
\]

- For a two-sided test, the p-value is:

\[
pval(T(x)) = \int_{-\infty}^{-|T(x) - \text{median}(T(X))|} Pr(T(x) \mid \theta = c) dT(x) + \int_{|T(x) - \text{median}(T(X))|}^{\infty} Pr(T(x) \mid \theta = c) dT(x)
\]

\[
pval(T(x)) = \sum_{\min(T(X))}^{-|T(x) - \text{median}(T(X))|} Pr(T(x) \mid \theta = c) + \sum_{|T(x) - \text{median}(T(X))|}^{\max(T(X))} Pr(T(x) \mid \theta = c)
\]
Review: Non-Intuitive Hypothesis Testing Concepts

- We do not know what the true model is (=parameter values are) in a real case!

- We assess a null hypothesis that we define!

- We assess this null hypothesis by calculating a p-value which assumes that the null hypothesis is true!

- We assess this null hypothesis by calculating a p-value from a single sample!

- We make one of two decisions: cannot reject or reject!
  - We decide on the value p-value that allows us to decide
  - If we reject, we interpret this as strong evidence against the null hypothesis being correct but we do not know for sure!
  - If we cannot reject, we cannot say anything (i.e., we have no evidence that the null is wrong and we cannot say that the null is right)!
Assume $H_0$ is correct (!): $\mu = 0$

**one-sided test**

- $p = 0.77$
- $T(x) = -0.755$

**Sample I:**
$T(x) = -0.755$

**two-sided test**

- $p = 0.45$
- $T(x) = 2.8$

**Sample II:**
$T(x) = 2.8$

$p = 0.0025$

**1**

$p = 0.005$

$\alpha = 0.05$

$C_\alpha = 1.64$

$\alpha = 0.05$

$C_\alpha = 1.96$

$H_0 : \mu = 0$

So, how do we make use of a p-value? Let's go back to our example case where we (see diagram on board for an example).

Also, note in this particular case:

1. We use the following formalism to write our null hypothesis $o$
2. We generally decide on some probability $p_{\text{value}}$ would be $\text{value}_p$
3. $k$ is the cumulative distribution function of $\mathcal{N}(\mu, \sigma^2)$
4. $\sigma$ is quite small. Can we interpret this as evidence against $H_0$ is quite arbitrary (and as we shall see, depends on $\delta$).

$C_\alpha$ is how we assess our null hypothesis. However, this is still does not provide us a guideline.
Results of hypothesis decisions I:
when H0 is correct (!!!)

- There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject

<table>
<thead>
<tr>
<th>Decision</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>cannot reject $H_0$</td>
<td>$H_0$ is true</td>
</tr>
<tr>
<td>reject $H_0$</td>
<td>$H_0$ is false</td>
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</tbody>
</table>

![Pr(T(x) | H0)](image-url)
Results of hypothesis decisions I: when H0 is correct (!!)

- There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject

<table>
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<th>Decision</th>
<th>Description</th>
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<tr>
<td>cannot reject H0</td>
<td>H₀ is true, 1-α, (correct)</td>
</tr>
<tr>
<td>reject H0</td>
<td>α, type I error</td>
</tr>
</tbody>
</table>

\[ \Pr(T(x) | H_0) \]

\[ T(x) \]

\[ c_\alpha = 1.64 \]
Results of hypothesis decisions I: when H0 is correct (!!)

- There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject

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<tr>
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</tr>
<tr>
<td>Reject $H_0$</td>
<td>$\alpha$, type I error</td>
</tr>
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There are two possible hypotheses:
- $H_0$ (null hypothesis): $\mu = \mu_0$
- $H_A$ (alternative hypothesis): $\mu \neq \mu_0$

- For a true $H_0$, the probability of observing a value of the test statistic $T(x)$ is $Pr(T(x) | H_0)$.
- For a true $H_A$, the probability of observing a value of the test statistic $T(x)$ is $Pr(T(x) | H_A)$.

Assuming $T(x)$ is normally distributed and $n=65$,

\[ Pr(T(x) | H_0) = N(0, 0.25) \]

\[ Pr(T(x) | H_A) = N(c_\alpha, 0.25) \]

For a significance level $\alpha = 0.05$, we choose $c_\alpha = 1.64$. Thus:

\[ Pr(T(x) > 2 | H_0) = Pr(T(x) > c_\alpha | H_0) = 0.05 \]

Now, in our example, we have considered a case where we reject the null hypothesis if the p-value is less than 0.05, which we use to define a p-value, which is a function of our statistic. If our p-value is less than 0.05 we reject $H_0$, where we do not know the specific values of the parameters. We experience that, if $H_0$ is correct (!!), then $Pr(T(x)) = Pr(T(x) > c_\alpha | H_0)$.

So, how do we make use of a p-value? Let's go back to our example case where we have a close relationship with p-values.

\[ pval = Pr(T(x) \leq 2 | H_0) = 1 - Pr(T(x) > 2 | H_0) = 0.95 \]

Can we interpret this as evidence against $H_0$? Yes we can, and intuitively, this corresponds to a specific value of $\alpha$ (see diagram on board for an example). Also, note in this particular case:

\[ pval = Pr(T(x) > 2 | H_0) = 0.05 \]

This lecture will complete our general discussion with the introduction of likelihood ratio tests.
Assume H0 is wrong (!): \( \mu = 3 \)

**Sample I:**
\( T(x) = -0.755 \)

**Sample II:**
\( T(x) = 2.8 \)
There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject.

Results of hypothesis decisions II: when H0 is wrong (!!)

- There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject.

<table>
<thead>
<tr>
<th>Decision</th>
<th>H0 is true</th>
<th>H0 is false</th>
</tr>
</thead>
<tbody>
<tr>
<td>cannot reject H0</td>
<td>1-(\alpha), (correct)</td>
<td>(\beta), type II error</td>
</tr>
<tr>
<td>reject H0</td>
<td>(\alpha), type I error</td>
<td>1 - (\beta), power (correct)</td>
</tr>
</tbody>
</table>

\[
\Pr(T(x) \mid H0)
\]

\[
x
\]

\[
\Pr(T(x) \mid H0)
\]
Results of hypothesis decisions II: when \( H_0 \) is wrong (!!!)

- There are only two possible decisions we can make as a result of our hypothesis test: reject or cannot reject

<table>
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<tr>
<th></th>
<th>( H_0 ) is true</th>
<th>( H_0 ) is false</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>cannot reject</strong> ( H_0 )</td>
<td>1-( \alpha ), (correct)</td>
<td>( \beta ), type II error</td>
</tr>
<tr>
<td><strong>reject</strong> ( H_0 )</td>
<td>( \alpha ), type I error</td>
<td>1 - ( \beta ), power (correct)</td>
</tr>
</tbody>
</table>

![Diagram showing hypothesis testing results](image.png)
Results of hypothesis decisions II: when $H_0$ is wrong (!!)

- There are only two possible decisions we can make as a result of our hypothesis test: *reject* or cannot *reject*

<table>
<thead>
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<th>Decision on $H_0$</th>
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<td>$\alpha$, type I error</td>
<td>$1 - \beta$, power (correct)</td>
</tr>
</tbody>
</table>

![Diagram showing the decision regions for hypothesis test](image)
Technical definitions

- Technically, correct decision given H0 is true is (for one-sided, similar for two-sided):
  \[ 1 - \alpha = \int_{-\infty}^{c_x} Pr(T(x)|\theta = c) dT(x) \]

- Type I error (H0 is true) is (for one-sided):
  \[ \alpha = \int_{c_x}^{\infty} Pr(T(x)|\theta = c) dT(x) \]

- Type II error given H0 is false is (for one-sided):
  \[ \beta = \int_{-\infty}^{c_x} Pr(T(x)|\theta) dT(x) \]

- Power is (for one-sided):
  \[ 1 - \beta = \int_{c_x}^{\infty} Pr(T(x)|\theta) dT(x) \]
Important concepts I

• REMEMBER (!!): there are two possible outcomes of a hypothesis test: we reject or we cannot reject
• We never know for sure whether we are right (!!!)
• If we cannot reject, this does not mean H0 is true (why?)
• Note that we can control the level of type I error because we decide on the value of $\alpha$
Important concepts II

- Unlike type I error $\alpha$, which we can set, we cannot control power directly (since it depends on the actual parameter value).
- However, since power $1 - \beta$ depends on how far the true value of parameter is from the $H_0$, we can make decisions to increase power depending on how we set up our experiment and test:
  - Greater sample size = greater power $1 - \beta$
  - Greater the value of $\alpha$ that we set = greater power $1 - \beta$ (trade-off!)
  - One-sided or two-sided test (which is more powerful?)
  - How we define our statistic (a more technical concept...)

$H_0$ (in general, cdfs of statistics $f$ see class for a diagram). We can also calculate the probability $Pr(x)$ for a two-tailed test for a case where we knew the true value of the parameters which are unknown to us.

$X \sim f(\mu, \sigma^2, \theta)$ is not correct. Where we set $x \sim f(\mu, \sigma^2, \theta)$ is not correct. Where we set $X \sim f(\mu, \sigma^2, \theta)$ it is quite arbitrary (and as we shall see, depends on the true value of $\mu$, $\sigma^2$ and $\theta$).

$H_0$ is true or $H_1$ is false. In general, for a one-sided test $H_0$ if we have $X = 1$ we can define a $T$ seeing 'yes' or 'no' when considering the question: is $X = 1$? Yes we can, and intuitively, this $H_0$ is going to be positive (or $H_1$ is correct).

However, since power $1 - \beta$ depends on how far the true value of parameter is from the $H_0$, we can make decisions to increase power depending on how we set up our experiment and test:

- Greater sample size = greater power $1 - \beta$

$Pr(x)$:

$H_0$ is true or $H_1$ is false. In general, for a one-sided test $H_0$ if we have $X = 1$ we can define a $T$ seeing 'yes' or 'no' when considering the question: is $X = 1$? Yes we can, and intuitively, this $H_0$ is going to be positive (or $H_1$ is correct).
That’s it for today

- Next lecture, we will begin our discussion of genetics!